Liebniz

Riley Moriss

November 19, 2025

After examining Descartes in 1637 it is amazing that the work of functions and geometry determined by equations could be carried so far so quickly as to the introduction of calculus in the same century. The wikipedia states that "Initially the work was not well received ... Only after the translation into Latin ... in 1649 did Descartes's masterpiece receive due recognition." So the timeline is even more impressive. In 1684 Liebniz published his first work [LIB14] on differential calculus, although there were many precursurs including Fermat and De Alambert [BM91].

The key thing to note is that the way we think about functions is very modern. We will need to examine what was considered and how they were approximated. At this point developments in notation are less exciting, most of the algebraic notation was already as we have it. It is well known what the Leibniz and Newton notation for calculus was, and we still use them today. We will therefore not focus on this as much as before.

Remark. We are drawing from [BB04]Here we want to talk about the converse of Descartes process, taking a geometric diagram and producing an algebraic equation, that seems to have been first outlined by Fermat and most clearly put in Wallis treatise on conics. Even though modern coordinates were not yet around, in Wallis we see a system of translating an algebraic equation into a curve. The equation would be relating the "ordinate" a line representing the modern y axis and the "abscissa" the x axis. In Wallis these are two lines that are positioned perpendicular and the equation is then interpreted as relating these two quantities in the modern way recalling that the relation that Descarte introduced was exactly that of length. $y = x^2$ is all the points where the length of the ordinate is equal to the square of the length of the abscissa line. So the same process as now, however the underlying referents are the actual geometric lines as opposed to the numbers. This is the context in which Liebniz understood analytic geometry.

Note that at this stage we are well and truly out of the Euclidean formalism. Curves in general are not covered by his axioms and no new ones are introduced. The curves are "the same sort of thing" as Euclids lines, however there are no rules on how they work. There are attempts at definitions, such as those curves given by this or that construction using this or that machine. On the algebraic side we have definitions and axioms for number theory that still apply in their domains and to some extent it seems like these are verified to be satisfied by the new science and so they transfer.

Remark. There is only scant translation of this seemingly important work. I have [Ste08], and then I used some LLMs to translate the linked Latin edition.

1 Functions

To talk about calculus in an era appropriate way we need to first establish the kinds of functions that might have been reasonably known about or differentiated. Now given that there was a sharp distinction between geometry and algebra until Descartes and even after that it was still not as straight forward as it is today, we need to be careful to destinguish what these functions were really doing.

First of all we have clearly seen in the work of Euclid, Diophantus and Descartes the use of *polynomial* functions. There are ancient attestations of many Greeks (and many more ancients from the east) working on trigonometric functions (for example Menelaus) and we have the treatise of Ptolemy which survives [PTG98]. After a very brief examination of [Kat07, 5.VII] we see that the islamic scholars saw sin "as a line and not a ratio", very much in the same veign as the Greeks. This line however in a Cartesian paradigm corresponds to a number (its length) and therefore opens up the question, can there be an algebraic equation for it? Finally logarithms and exponentials seem to have been a burgeoning area of investigation in the 1600's with Napier in 1614 an Huygens in 1661 talking about them respectively (according to wiki).

Perhaps this analysis is wrongheaded as Lieniz will want to differntiate *curves* not functions. From [Des54] we have the following quote too

> The term, mechanical curve, was used by Descartes to designate a curve that cannot be expressed by an algebraic equation." Leibniz and others call them transcendental.

Which reminds us that the ancients as well as Descartes also considered things like spirals etc that didnt seem to have simple constructions as curves, however of a mechanical (physical) variety.

All of this is to say that perhaps at the time of Liebniz the only reasonable functions to differentiate, those that corresponded under Descartes scheme to curves in the plane, would have been something like what we now call rational functions, ratios of polynomials. Other functions were known about however it was not as clear how to represent them *algebraically*, and so they could not themselves coorespond to curves in the plane in a straigtforward way, thus preventing the tangential analysis.

Remark. Here are some interesting things to note from our inspection of Ptolemys work: Ptolymy wrote in Alexandria in around 150 AD. It brings to the surface a subtlety that we have been able to ignore so far, Euclid defines

> And a plane angle is the inclination of the lines to one another, when two lines in a plane meet one another, and are not lying in a straight-line.

this seems to again teeter on the edge of the geometry-arithmetic dualism. Proclus (400 AD) puts the situation "angle must be either a quality, a quantity, or a relationship." Whatever it may be, already in Ptolemy they were "dividing the circumference into 360 parts". Between any two points of the circle you can get a chord, that is a straight line, moreover from the center of the circle there is a triangle joining the two ends of the chord, the angle in the center is called the angle "subtended" by the chord. Ptolomy then records the "ratio" of these chords to the diameter. In modern language if we considered one of the sides of the triangle other than the chord itself then this is exactly a calculuation of the "magnitude" of the chord from the given angle. How this *can* be connected to modern notation is explained here, however it was not done.

2 The First Paper of Liebniz

Here we will review [LIB14] wich is a mere 8 pages. L immediately begins by defining his coordinates in terms of a fixed line "the axis" and then referes to the "ordinates normal to the axis" and the "abscissa for the axis". The ordinates of his curves are given the variable names u, v, w etc. and he defines his "differentials" or "differences" as

> Now some right line taken arbitrarily may be called dx

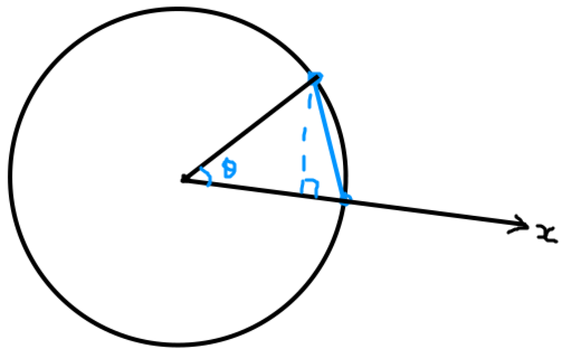

Figure 1: The chord in blue.

(right to the axis). He later changes his perspective on the differentials

> dx themselves can be had as proportional differences of x either with momentary increments or decrements.

Likewise for dv, du, dw etc. By this we are interpretting that we can replace the concept of the line with the idea that we have now of $dx \simeq x_1 - x_2$, for two close values of x_i. With this notation he immediately launches into a shockingly modern treatment of the differential:

> Let a be a given constant quantity, $\mathrm{d}a$ will be equal to 0, and $\mathrm{d}(ax)$ will be equal to $a\mathrm{d}x$.
>
> Now for Addition and Subtraction: $\mathrm{d}(z - y + w + x$ shall be equal to $\mathrm{d}z - \mathrm{d}y + \mathrm{d}w + \mathrm{d}x$.
>
> Multiplication: $\mathrm{d}(xy) = x\mathrm{d}y + y\mathrm{d}x$

He also deals in the same way with roots, quotientsOn the use of the differential he remarks ”It is required to be noted both x and dx are to be treated in the same manner in this calculation” by which we think he is clarifying that in the same way that x is treated algebraically so is dx. He then exposits the meaning of negative quantities showing up in these calculuations, apparently still a source of concern, as drawing the line in the opposite direction or else a decrease in the rate (that is *relative* to the given coordinates increase). He then classifies the types of turning points (inflection, concavity etc).

L prescribes that ”it arises, that it shall be possible for any proposed equation to write the equation of its differential” by following the above rules algorithmically applying them to an equation. Note that he claims

> it is apparent our method extends to transcendental lines

which means that we cannot take his algebraic statements as a *definition*, as these clearly do not extend to transcendental lines (those that are not algebraically expressable, also called mechanical lines).

3 Philosphy

A radical simplification of the shift we see occuring is as follows: After Descarte unified geometry and algebra the new relevant issue at hand was the infinite and the curved. Making these limiting arguments rigerous. There is no foundation in Euclid, Descartes or Lienbiz for curves. Of course Euclid defined them, however he has no postualtes for them and therefore no way of constructing them. Descarte addresses this by saying that his more complex curves can be given by more complex machines than ruler and compass, however he neglects to formalise this (he claims that his curves are no less rigeorus than the ancients just because the machines are more complex, however I claim they are less rigerous for their lack of stated postulates). Already at the time of Descartes notation is somewhat settled, and even more so with the work of Euler. With Liebniz we embark on the new era of mathematics, the era of analysis.

Unlike Euclidean geometry or number theory the field of analysis does not start with axioms or postulates but physical intuition. From its conception in Descartes treatise on philosophy, physics and biology it is tied to the intuitions of men about nature. It would only become on issue *later* to formalise it. Apparently in the late 1700's with Cauchy this was pushed forward.

We can also see a tension already in Liebniz between curve and function, between curves and algebra still. For instance what does "let a be a given constant quantity" mean? a is a line? or a curve with a constant abscissa, that is a straight line? In which case it is not clear why it should be zero and not the line itself? Im not sure what is mean by this in Liebniz, however it seems as though the concept of function (apparently introduced by Euler) and real number could not come soon enough to clarify the situation.

Note also that this renewed interest in mechnical or transcendental curves motivates the modern definition of the real numbers as including *transcendental* numbers, and not just say constructable numbers (as it would have if we only dealt with say numbers arrising in Euclidean geometry).

From here on out we will want to pay closer attention to when people actually make the move from lines to *numbers and functions*. I think that the discussion around infinity is a bit played out and so I probably wont take much interest in it. Attempts at putting in axioms for this new geometry and any further clarifications that might take place are also what we will look out for.

References

[BB04] Carl Benjamin Boyer and Carl Benjamin Boyer. *History of analytic geometry*. Dover books on mathematics. Dover Publications, Mineola, NY, dover ed edition, 2004.

[BM91] Carl Benjamin Boyer and Uta C. Merzbach. *A history of mathematics*. Wiley, New York, 2. ed edition, 1991.

[Des54] René Descartes. *The Geometry of René Descartes: With a Facsimile of the First Edition*. Dover Books on Mathematics Series. Dover Publications, Incorporated, Newburyport, 1st ed edition, 1954.

[Kat07] Victor J. Katz. *The mathematics of Egypt, Mesopotamia, China, India, and Islam: a sourcebook*. Princeton university press, Princeton (N.J.), 2007.

[LIB14] G.W. LEIBNIZ and (trans) Ian Bruce. A NEW METHOD FOR FINDING MAXIMA AND MINIMA. 2014.

[PTG98] Claudius Ptolemaeus, Gerald J. Toomer, and Owen Gingerich. *Ptolemy's Almagest*. Princeton University Press, Princeton, 1998.

[Ste08] Jacqueline A. Stedall. *Mathematics Emerging: A Sourcebook 1540-1900*. Oxford University Press USA - OSO, Oxford, 2008.